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COMMENT

# Infinite AB percolation clusters exist

Edward R Scheinerman and John C Wierman

Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, MD 21218, USA

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**Abstract.** We provide an example of a lattice on which infinite AB percolation clusters can occur, answering a question of Halley. We identify parameter values for which infinite AB percolation occurs and does not occur on our lattice.

Halley (1983) considered a variant of the percolation model, called AB percolation, which is motivated by species bonding considerations. Two types of species, labelled A and B, randomly occupy the sites of an infinite lattice graph $G$, with probabilities $p$ and $1-p$ respectively. Neighbouring species of opposite type are bonded together, while species of the same type do not bond. The object of study is the size distribution of clusters of bonded species (AB clusters).

Halley proved that, if the graph $G$ is bipartite and has a site percolation critical probability strictly greater than one-half, then there are almost surely no infinite AB clusters when $p=\frac{1}{2}$. Since, intuitively, one expects that the probability of an infinite AB cluster is largest at $p=\frac{1}{2}$, this suggests that infinite AB percolation does not occur on such graphs for any value of $p$. A mathematically rigorous proof of the non-existence of AB percolation on a subclass of such graphs is presented in Appel and Wierman (1986).

Halley (1986) stated that the existence of infinite AB percolation has not been proven for any two-dimensional lattice graph $G$. The set of parameter values for which infinite AB percolation clusters exist must be symmetric about $p=\frac{1}{2}$, but has not been proven to consist of a single interval. Monte Carlo simulations of Mai and Halley (1981) suggest that infinite AB cluster occurs for $p\in[0.2145, 0.7855]$ on the triangular lattice. Current research by Appel and Wierman promises to prove the existence of AB percolation on the triangular lattice and identify the interval (for the parameter $p$) on which infinite AB percolation clusters almost surely exist. The purpose of this comment is to construct a simple lattice graph, based on the square lattice, on which infinite AB percolation exists with positive probability for an interval of $p$ values.

Let $G$ be the graph whose site set is all lattice points in the plane, i.e. all points both of whose coordinates are integers. We join two sites by a bond iff their distance is at most 2. Every site is joined to 12 other sites; for example, the origin $(0,0)$ is joined to $(\pm1,0)$, $(0,\pm1)$, $(\pm1,\pm1)$, $(\pm2,0)$ and $(0,\pm2)$.

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We now consider a second graph $H$ which is simply the square lattice. Each site of $H$ corresponds to four sites of $G$:

$$(i, j) \rightarrow S_{ij} = \{(2i, 2j), (2i+1, 2j), (2i, 2j+1), (2i+1, 2j+1)\}.$$

Every pair of sites in $S_{ij}$ are adjacent, so if $S_{ij}$ contains both an A and a B site, all sites in $S_{ij}$ are in a common AB cluster.

Moreover, suppose $(i, j)$ and $(m, n)$ are adjacent sites of the square lattice $H$ (for example, $m = i$ and $n = j+1$). Thus the sites of $G$ in $S_{ij} \cup S_{mn}$ form a $4 \times 2$ grid. One checks that if both $S_{ij}$ and $S_{mn}$ contain both an A and a B site, then all sites in $S_{ij} \cup S_{mn}$ are in the same AB cluster.

Suppose the sites in $G$ are labelled A or B independently with probability $p = \frac{1}{2}$. We declare a site $(i, j)$ in $H$ open if and only if $S_{ij}$ contains *both* an A and a B site; otherwise $(i, j)$ is closed. If $(i, j)$ and $(m, n)$ are adjacent, open sites of $H$, then all sites in $S_{ij} \cup S_{mn}$ are in a common AB cluster. Observe that the sites of $H$ are open with probability $\frac{7}{8}$ and closed with probability $\frac{1}{8}$. Since the $S_{ij}$ are disjoint, the sites are open or closed independently of one another. Since $\frac{7}{8}$ is greater than the percolation threshold for the square lattice (see Hammersley (1959) where it is shown that the critical probability is less than 0.65), we know there exist infinite open clusters in $H$. This implies, by the earlier remarks, infinite AB clusters in $G$.

(Clearly, by considering values of $p$ other than $\frac{1}{2}$, one may determine an interval on which AB percolation occurs. However, the technique here is not appropriate for determining the exact range of values for which AB percolation occurs.)

Note, however, that not all values of $p$ result in AB percolation. Let $H^*$ be the matching lattice of the square lattice (see Sykes and Essam 1964). The sites of $H^*$ are the lattice points in the plane. Two sites of $H^*$ are joined iff they are within a distance $\sqrt{2}$ of one another.

We consider a modified correspondence between $H^*$ and $G$ which is similar to the previous correspondence between $H$ and $G$. Let $T_{ij}$ denote a $4 \times 4$ set of sites of $G$, specifically:

$$T_{ij} = \{(4i + \alpha, 4j + \beta) : 0 \leqslant \alpha, \beta \leqslant 3\}.$$

Each site $(i, j)$ of $H^*$ will be associated with the $4 \times 4$ set $T_{ij}$.

We call a site $(i, j)$ in $H^*$ open iff $T_{ij}$ contains at least one B site; otherwise (when all sites of $T_{ij}$ are A) the site $(i, j)$ is called closed.

Recall that $p$ is the probability that a site of $G$ is A (and $1-p$ it is B). For each $(i, j)$ we see that the probability that all sites in $T_{ij}$ are A is $p^{16}$ which we can take arbitrarily close to 1 (put $p = (1-\varepsilon)^{1/16}$). Therefore, the probability a site of $H^*$ is open equals $1 - p^{16}$ (which is nearly 0 and therefore below the critical probability for $H^*$). Thus the probability the origin of $H^*$ is contained in an infinite open cluster is 0. By Sykes and Essam (1964) this is equivalent to the almost sure existence of a circuit of $H^*$ encircling the origin, all of whose sites are closed and in which adjacent sites are distance 1 (i.e. the circuit only uses horizontal and vertical bonds).

What are the implications of this closed circuit for $G$? Observe that this implies that the probability that the origin (or any specific site) of $G$ is contained in an infinite AB cluster is 0 since this 'thick' closed circuit around the origin does not allow penetration by an AB cluster. The event that some infinite AB cluster exists is a countable union of events (existence of an infinite AB cluster containing a specific site) each of which has probability 0. Hence for $p$ sufficiently close to 1 (or 0), we do not have infinite AB percolation clusters in $G$.

We have shown the existence of an interval for the parameter $p$ containing $\frac{1}{2}$ in which infinite AB percolation almost surely occurs, as well as intervals containing 0 and 1 in which infinite AB percolation almost surely does not occur. We cannot rule out, however, the possibility of multiple thresholds between 0 and $\frac{1}{2}$; the set of $p$ values for which infinite AB clusters almost surely exist might fail to be connected.

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